5. GERTSENSHTEIN S.YA. and SHTEMLER YU.M., On perturbations of finite amplitude in the boundary layer. Izv. Akad. Nauk SSSR, MZhG, 1, 1976.
6. SMITH F.T., Non-linear stability of boundary layers for disturbances of various sizes. Proc. Roy. Sos. London. Ser. A, 368, 1735, 1979.
7. RUBAN A.I., The non-linear equation for the amplitude of the Tollmin-Schlichting wave in a boundary layer. Izv. Akad. Nauk SSSR, MZhG, 6, 1983.
8. RYZHOV O.S. and TERENT'YEV E.D., On the non-stationary boundary layer with selfinduced pressure. PMM, 41, 6, 1977.
9. TERENT'YEV E.D., The linear problem of a vibrator in a subsonic boundary layer. PMM, 45, 6, 1981.
10. ZHUK V.I. and RYZHOV O.S., On the asymptotic form of solutions of the Orr-Sommerfeld equations describing unsteady oscillations at large Reynolds numbers. Dokl. Akad. Nauk SSSR, 268, 6, 1983.
11. MIKHAILOV V.V., On the asymptotic form of neutral curves of the linear problem of the stability of the laminar boundary layer. Izv. Akad. Nauk SSSR, MZhG, 5, 1981.

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ON FORCED OSCILLATIONS IN THE BOUNDARY LAYER AT FREQUENCIES NEAR THE UPPER BRANCH OF THE NEUTRAL CURVE*

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Perturbations introduced into the boundary layer (BL) of an incompressible liquid by a harmonic oscillator in the form of a moving section of the surface are examined. Outside the oscillating part, the streamlined solid is a plane plate. It is assumed that the Reynolds number is large and the oscillation frequency, corresponds, in order of magnitude, to the asymptotic form of the upper branch of the neutral stability curve (NSC). The system of equations for perturbations, at small amplitudes of the oscillator, is linearized and is solved by the Fourier method. In addition, for each Fourier component, the flow field is divided into five sublayers. The amplitude of a Tollmin-Schlichting wave (TS) is calculated and separated from the perturbed background downstream of the oscillator. If the oscillator frequency exceeds the neutral value at the upper branch of NSC with the given Reynolds number, the TS wave amplitude decays. For frequencies below neutral, the wave amplitude increases exponentially downstream. In the final example, the parameters of the TS wave fall within the unstable region, between two NSC branches.

At a distance L^* from the front edge of the plane surface with an incompressible viscous liquid flowing over it, let there be a moving section of surface of length l^* which is oscillating at a frequency ω^* (henceforth the asterisk denotes dimensional quantities). Defining the Reynolds number as $R = U_\infty{}^*L^*/\nu^*$, where $U_\infty{}^*$ is the velocity of the oncoming flow and ν^* is the kinematic viscosity, we will assume $R \to \infty$. An investigation of the perturbation propagation process caused by the moving section is one of the problems of BL reproducibility.

The solution of this problem taking compressibility into account and for any Mach number at infinity was obtained previously in /1, 2/ for $l^* = O(R^{-3/8}L^*)$, $\omega^* = O(R^{1/4}U_\infty{}^*L^{*-1})$, starting from the linearized equations of the theory of free interaction /3, 4/, which, as is well-known /5, 6/, is subject to perturbations in the given range of frequencies. In the subsonic case, the first mode from the spectrum of eigensolutions of these equations corresponds to the

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limiting form of TS wave for large Reynolds numbers in the neighbourhood of the lower branch of the NSC. As follows from /2, 7/, perturbations downstream from the oscillator degenerate into precisely this wave.

Below, the question of TS wave generation in proximity to the upper branch of the NSC is examined. Naturally, the orders of the frequency and of the characteristic length differ from those assumed in the theory of free interaction, namely $l^* = O(R^{-9/20}L^*)$ and $\omega^* = O(R^{2/5}U_\infty^*L^{*-1})$ are assumed. We put $\sigma = R^{-1/20}$. The mobile part of the surface is specified by the equation

$$y_w^* = h\sigma^{14}L^*G(T, X),\ T = \sigma^{-8}U_\infty^*L^{*-1}t^*,\ X = \sigma^{-9}L^{*-1}(x^* - L^*) \tag{1}$$

where t^* is the time, x^*, y^* are the coordinates of a Cartesian system with the x^* axis directed along the plate around which the flow occurs, and h is a dimensionless parameter.

First, we note that, irrespective of the causes of oscillations of the fluid in the BL, there is a viscous sublayer in the vicinity of the wall with thickness of order of $R^{-1/2}L^*(\omega^*L^*/U_\infty^*)^{-1/2} = \sigma^{14}L^*$. In particular, this type of sublayer exists in the case of free oscillations, when $G = 0$. It can be seen from Eq.(1) that, for $h = O(1)$, the part of the surface which changes shape sinks into the viscous sublayer caused, in this instance, by oscillations of this part itself.

We will find the orders of the perturbations of the components u^* and v^* of the velocity vector and the pressure p^* for flows round an oscillating non-uniformity. The surface of the moving part on the plate has the vertical component of velocity $v_w^* = h\sigma^6U_\infty^*[\partial G/\partial T + O(\sigma^{10})]$, whereas the tangential component of the surface velocity, of the order of $h\sigma^{11}U_\infty^*$, can be neglected. The components $U_\infty^*U_0, U_\infty^*V_0$ of the velocity vector of the unperturbed steady flow in the BL on a plane plate can be represented in the form

$$U_0 = U_B(x, Y_m) + O(\sigma^{10}),\ V_0 = \sigma^{10}V_B(x, Y_m) + O(\sigma^{20}) \tag{2}$$
$$x = L^{*-1}x^*,\ Y_m = \sigma^{-10}L^{*-1}y^*$$

where the functions U_B and V_B, for the first of which the following limit properties hold,

$$Y_m \to 0,\ U_B \to \lambda_1(x)Y_m + \lambda_4(x)Y_m^4/24 + \ldots \tag{3}$$
$$Y_m \to \infty,\ U_B \to 1 + \ldots$$

are specified by the Blasius solution. As a result of the deformation of the wall, the adhesion condition must be satisfied, not for $Y_m = 0$, but on the contour $Y_m = h\sigma^4G$, where, by virtue of (3), the unperturbed velocity is of the order of $h\sigma^4U_\infty^*$. Therefore, the perturbation of the longitudinal component of the velocity u^* is of the same order. If this last estimate is also true for flows round a stationary non-uniformity, then the perturbation of the vertical velocity component v^* of order $h\sigma^6U_\infty^*$ is linked to the change in the shape of the non-uniformity with time. The change in the pressure p^* of a fluid of density ρ^*, compared with the value p_∞^* in a flow from infinity using the equation of conservation of the component of momentum along the x^* axis is estimated as $O(h\sigma^5\rho^*U_\infty^{*2})$.

These estimates, obtained in the region of the wall in the vicinity of the oscillator are also correct for all BL thicknesses, with the exception of the perturbation of the vertical velocity component. In fact, for $Y_m = O(1)$, it follows from the equation of continuity that the perturbation v^* is of the order of $h\sigma^5U_\infty^*$.

In the base part of the BL the field of flow parameters can be written in the form

$$\left(\frac{u^*}{U_\infty^*}, \frac{v^*}{U_\infty^*}, \frac{p^* - p_\infty^*}{\rho^*U_\infty^{*2}}\right) = (U_0 + h\sigma^4u_m, V_0 + h\sigma^5v_m, P_0 + h\sigma^5p_m) \tag{4}$$

where $P_0 = O(\sigma^{10})$ is the unperturbed pressure, U_0, V_0, P_0 are functions of x, Y_m and σ, while u_m, v_m and p_m are functions of T, X, Y_m, σ and h. The Navier-Stokes equations are transformed in the following way, as a result of substituting expressions (4) into them, taking relations (2) into account:

$$\sigma\frac{\partial u_m}{\partial t} + U_B\frac{\partial u_m}{\partial X} + h\sigma^4u_m\frac{\partial u_m}{\partial X} + \sigma^9u_m\frac{\partial U_B}{\partial x} + v_m\frac{\partial U_B}{\partial Y_m} + \tag{5}$$
$$h\sigma^4v_m\frac{\partial u_m}{\partial Y_m} + \sigma^9V_B\frac{\partial u_m}{\partial Y_m} = -\sigma\frac{\partial p_m}{\partial X} + \sigma^9\left(\frac{\partial^2u_m}{\partial Y_m^2} + \sigma^2\frac{\partial^2u_m}{\partial X^2}\right)$$
$$\sigma^2\frac{\partial v_m}{\partial T} + \sigma U_B\frac{\partial v_m}{\partial X} + h\sigma^5u_m\frac{\partial v_m}{\partial X} + \sigma^{10}v_m\frac{\partial V_B}{\partial Y_m} + h\sigma^5v_m\frac{\partial v_m}{\partial Y_m} +$$
$$\sigma^{10}V_B\frac{\partial v_m}{\partial Y_m} = -\frac{\partial p_m}{\partial Y_m} + \sigma^{10}\left(\frac{\partial^2v_m}{\partial Y_m^2} + \sigma^2\frac{\partial^2v_m}{\partial X^2}\right), \qquad \frac{\partial u_m}{\partial X} + \frac{\partial v_m}{\partial Y_m} = 0$$

As may be seen from (5), non-parellelism of the BL distorts functions with subscript m by an amount of the order of σ^9. The non-linear terms make a contribution of the order of $h\sigma^4$. The relative value of the viscous terms in Eqs.(5), generally speaking, can exceed the value of σ^9. For example, the viscous terms are essential in the critical layer (for travelling-wave type perturbations) and also in proximity to the surface around which flow occurs.

Equality of the orders of the term with the second derivative along the vertical coordinate and that of the non-stationary term in the first equation of system (5) is achieved if $Y_m = \sigma^4 Y_l$, $Y_l = O(1)$. The new variable $Y_l = \sigma^{-14} L^{*-1} y^*$ of the viscous wall sublayer in which the oscillator is submerged is introduced by the last relation. In accordance with the estimates in the wall sublayer introduced above we have

$$\left(\frac{u^*}{U_\infty^*}, \frac{v^*}{U_\infty^*}, \frac{p^* - p_\infty^*}{\rho^* U_\infty^{*2}}\right) = (U_0 + h\sigma^4 u_l, V_0 + h\sigma^8 v_l, P_0 + h\sigma^5 p_l) \tag{6}$$

where U_0, V_0, P_0 depend on $x, \sigma^4 Y_l$ and σ, while u_l, v_l and p_l depend on T, X, Y_l, σ and h. Moreover, the unperturbed profile of the longitudinal component of the velocity is given by the expression $U_0 = \sigma^4 \lambda_1 Y_l + O(\sigma^{16})$. From the Navier-Stokes equations we have

$$\begin{gathered}\frac{\partial u_l}{\partial T} + \sigma^3 \lambda_1 Y_l \frac{\partial u_l}{\partial X} + \sigma^3 h u_l \frac{\partial u_l}{\partial X} + \lambda_1 v_l + h v_l \frac{\partial u_l}{\partial Y_l} = \\ -\frac{\partial p_l}{\partial X} + \frac{\partial^2 u_l}{\partial Y_l^2} + O(\sigma^8), \quad \frac{\partial p_l}{\partial Y_l} = O(\sigma^7), \quad \sigma^3 \frac{\partial u_l}{\partial X} + \frac{\partial v_l}{\partial Y_l} = 0\end{gathered} \tag{7}$$

The system of Eqs.(7) may be derived from system (5), by changing to the variable Y_l and using relations (3).

Boundary conditions on the surface take the form

$$Y_l = hG, \quad u_l = -\lambda_1 G, \quad v_l = \partial G/\partial T \tag{8}$$

The dependence on the "slow" coordinate of x of the functions U_B and V_B as also the coefficients λ_1, λ_4 at distances $X = O(1)$ change them by an amount of the order of σ^9, Therefore, everywhere below we assume $x = 1$ and, in particular, $\lambda_1 = 0.3321$, $\lambda_4 = -\lambda_1^2/2$.

Let us omit in Eqs.(7) the correction terms denoted by the symbol O of higher order of smallness with respect to the parameter σ, while retaining terms of the order of σ^3. To linearize the first equation of system (7) we will assume the parameter h to be small. Let us consider the case $h \ll \sigma^3$. Let the wall oscillate periodically with time

$$G(T, X) = G'(X) \exp(i\Omega T)$$

at a frequency whose value in the initial system of units is $\omega^* = \sigma^{-8} U_\infty^* L^{*-1} \Omega$. We will represent the required flow functions in the lower sublayer as

$$(u_l, v_l, p_l) = (u_l', v_l', p_l') \exp(i\Omega T) \tag{9}$$

We pass from primed amplitudes to their Fourier transform which is denoted by a bar, for example

$$\bar{u}_l(\Omega, K, Y_l, \sigma) = \int_{-\infty}^{\infty} u_l'(\Omega, X, Y_l, \sigma) \exp(-iKX)\, dX$$

Eqs.(7) with their linear terms neglected give

$$\begin{gathered}i(\Omega + \sigma^3 \lambda_1 K Y_l)\bar{u}_l + \lambda_1 \bar{v}_l = -iK\bar{p}_l + \partial^2 \bar{u}_l / \partial Y_l^2 \\ \partial \bar{p}_l / \partial Y_l = 0, \quad i\sigma^3 K \bar{u}_l + \partial \bar{v}_l / \partial Y_l = 0\end{gathered} \tag{10}$$

The boundary adhesion conditions were transferred on to the line $Y_l = 0$, which allows of the same error as for the neglected non-linear terms. We then obtain from (8)

$$Y_l = 0, \quad \bar{u}_l = -\lambda_1 \bar{G}, \quad \bar{v}_l = i\Omega \bar{G} \tag{11}$$

With the help of (10) and (11) we can express the functions $\bar{u}_l, \bar{v}_l$ in terms of the previously unknown Fourier transform of the pressure $\bar{p}_l$. In the plane of the complex variable

$$\begin{gathered}z_l = \sigma(i\lambda_1 K)^{1/3} Y_l + \zeta_l, \quad \zeta_l = \sigma^{-2} i^{1/3} \Omega (\lambda_1 K)^{-2/3}, \\ -3\pi/2 < \arg K < \pi/2\end{gathered}$$

we have

$$\frac{\partial^3 \bar{u}_l}{\partial z_l^3} - z_l \frac{\partial \bar{u}_l}{\partial z_l} = 0, \quad \bar{v}_l = i\Omega \bar{G} - \sigma^2 \lambda_1^{-1/3} (iK)^{2/3} \int_{\zeta_l}^{z_l} \bar{u}_l\, dz_l \tag{12}$$

$$z_l = \zeta_l, \quad \frac{\partial^2 \bar{u}_l}{\partial z_l^2} = \sigma^{-2} \lambda_1^{-2/3} (iK)^{1/3} \bar{p}_l, \quad \bar{u}_l = -\lambda_1 \bar{G}$$

The solution of problem (12), which satisfies the condition for there to be no exponential growth at infinity (along the radius $\arg z_l = \pm\pi/6$), is given by the expression

$$\bar{v}_l = i\Omega\bar{G} + \sigma^2 (i\lambda_1 K)^{1/3} \bar{G} (z_l - \zeta_l) - \frac{iK\bar{p}_l}{\lambda_1} \left[\frac{d\,\mathrm{Ai}(\zeta_l)}{\partial z_l}\right]^{-1} \left[z_l \int_{\zeta_l}^{z_l} \mathrm{Ai}(z)\,dz - \frac{d\,\mathrm{Ai}(z_l)}{dz_l} + \frac{d\,\mathrm{Ai}(\zeta_l)}{dz_l}\right] \tag{13}$$

Here, $\mathrm{Ai}(z_l)$ is the Airy function. Since $|z_l|$ and $|\zeta_l|$ are of the order of σ^{-2}, we replace the Airy function in (13) by its asymptotic form /8/ for large values of the argument. Returning to the variable Y_l, we obtain

$$\begin{aligned} \bar{u}_l &= -\lambda_1\bar{G} + K\Omega^{-1}\bar{p}_l\{\exp[-(i\Omega)^{1/2} Y_l] - 1\} + \sigma^3\lambda_1 K^2 (i\Omega)^{-5/2} \bar{p}_l \times \\ &\quad \{\exp[-(i\Omega)^{1/2} Y_l][1 + {}^3/_4 (i\Omega)^{1/2} Y_l + {}^1/_4 i\Omega Y_l^2] - 1\} + O(\sigma^6) \\ \bar{v}_l &= i\Omega\bar{G} + i\sigma^3\lambda_1 K\bar{G} Y_l + \sigma^3 K^2 (i\Omega)^{-3/2} \bar{p}_l \{1 - (i\Omega)^{1/2} Y_l - \\ &\quad \exp[-(i\Omega)^{1/2} Y_l]\} + O(\sigma^6); \quad \arg\Omega = \begin{cases} 0, & \Omega > 0 \\ -\pi, & \Omega < 0 \end{cases} \end{aligned} \tag{14}$$

We will now construct a solution in the basic thickness of the BL. We will neglect non-linear terms in (5) and also terms connected with the non-parallelism of the initial flow in the BL. Separating the time factor $\exp(i\Omega T)$, as in (9), in all required functions with subscript m and carrying out a Fourier transformation along the X coordinate, we reduce system (5) to the Orr-Sommerfeld equation

$$(U_B - c)\left(\frac{\partial^2\bar{v}_m}{\partial Y_m^2} - \delta^2\bar{v}_m\right) - \bar{v}_m \frac{\partial^2 U_B}{\partial Y_m^2} = \frac{\sigma^9}{iK}\left(\frac{\partial^4\bar{v}_m}{\partial Y_m^4} - 2\delta^2\frac{\partial^2\bar{v}_m}{\partial Y_m^2} + \delta^4\bar{v}_m\right), \quad c = -\frac{\sigma\Omega}{K}, \quad \delta = \sigma K \tag{15}$$

where $\bar{v}_m$ is the Fourier transform of the vertical component of the velocity. The parameter c in Eq.(15) relays the role of the phase velocity, since every Fourier perturbation component with a periodic time dependence is a travelling wave. The parameter δ has the meaning of the ratio of the BL thickness to the wavelength of the perturbations.

The viscous terms on the right-hand side of Eq.(15), which are of the order of the neglected terms arising from the growth of the BL thickness, must also be omitted everywhere, with the exception of the critical layer. The value of the coordinate $Y_m = Y_m^c$, in the neighbourhood of which the critical layer is situated, corresponds to the equality $U_B = c$. Moreover, as a result of the smallness of $c = O(\sigma)$ using expansion (3) we find $Y_m^c = \lambda_1^{-1}c - \lambda_4\lambda_1^{-4}c^4/24$. It is known that when there are no viscous terms, the solution of Eq.(15) at the point $Y_m = Y_m^c$ has a logarithmic singularity. By equating the left- and right-hand sides of Eq.(15) in this singular solution (derived below) we obtain an estimate of the thickness of the critical layer $Y_m - Y_m^c = O(\sigma^3)$.

In the inviscid Eq.(15), omitting the right-hand side, the parameters c and δ are conveniently considered as independent and can be represented in the form /9, 10/

$$\bar{v}_m = f_0 + \delta^2 f_1 + \ldots \tag{16}$$

assuming the value of c to be fixed. To a first approximation

$$(U_B - c)\frac{\partial^2 f_0}{\partial Y_m^2} - \frac{\partial^2 U_B}{\partial Y_m^2} f_0 = 0 \tag{17}$$

Outside the BL, where $U_B = 1$, we select the exponential decaying solution of the form $a\exp(\mp Y_u)$, $Y_u = \delta Y_m$ as $Y_u \to \infty$ from the two linearly independent non-viscous solutions of (15). Assuming $Y_u \to 0$ we obtain the asymptotic boundary condition on the external boundary of the BL

$$Y_m \to \infty, \quad \bar{v}_m \to a(1 \mp \delta Y_m + {}^1/_2\delta^2 Y_m^2 + \ldots) \tag{18}$$

Here and henceforth the upper symbol corresponds to $K > 0$, the lower to $K < 0$, and the constant a is unknown. The limit expression (18) shows that f_0, f_1 may depend on δ in view of the boundary conditions. Of course, this does not make expansion (16) incorrect.

The solution of Eq.(17), satisfying the condition $f_0 \to a(1 \mp \delta Y_m)$ where $Y_m \to \infty$, is written in the following way:

$$f_0 = a(1-c)^{-1}(U_B - c) \pm \delta a(1-c)(U_B - c)\times \left\{ \int\limits_{Y_m}^{\infty} \left[\frac{1}{(U_B - c)^2} - \frac{1}{(1-c)^2} \right] dY_m' - \frac{Y_m}{(1-c)^2} \right\} \tag{19}$$

Expression (19) is suitable for all real Y_m if $c < 0$. If $c > 0$ the expression only has meaning for $Y_m > Y_m^c$. The asymptotic form of the solution in the critical layer at its lower boundary gives the condition for matching, by which the solution of the non-viscous Eq.(17) between the critical layer and the wall is unique. This asymptotic form can be computed even without reference to the solution in the critical layer itself. Indeed, as follows from /9, 10/, when $K > 0$ the asymptotic expression for the viscous solution on the upper boundary of the critical layer permits an analytical continuation at its lower boundary, through the region $\sigma^3 \ll |Y_m - Y_m^c| \ll c$ of complex values of Y_m, circumventing the point Y_m^c in a clockwise direction. Similarly, when $K < 0$ the point Y_m^c is bypassed anticlockwise. But the asymptotic form on the upper boundary of the critical layer, however, is known and is set by the limit of expression (19) where $Y_m = Y_m^c + 0$. Hence it follows that the function (19) is a solution of the non-viscous Eq.(17) as well as in the region between the critical layer and the wall (for real $0 < Y_m < Y_m^c$), if the interval in (19) is taken along a path which goes round the singular point Y_m^c according to the rule described above.

Let $Y_m = O(c)$. Then expression (19) can be transformed to the form

$$f_0 = a\lambda_1(Y_m - Y_m^c) \pm \frac{a\delta}{\lambda_1} \pm \frac{a\lambda_4}{2\lambda_1^3}\delta c^2 \theta(c)(Y_m - Y_m^c)\times [\ln|Y_m - Y_m^c| + i\arg(Y_m - Y_m^c)] + \ldots \tag{20}$$

where for real Y_m we must put $\arg(Y_m - Y_m^c) = 0$, if $Y_m > Y_m^c$, and $\arg(Y_m - Y_m^c) = -\pi \operatorname{sign} K$, if $Y_m < Y_m^c$; $\theta(c)$ is the Heaviside unit function.

We will match the components of the velocity vector on the upper boundary of the viscous wall sublayer, where $\sigma^4 \ll Y_m \ll c$. Changing to the "inner" variable $Y_l = \sigma^{-4}Y_m$ in (20) we obtain

$$\begin{aligned} \bar{u}_m &= ia\left[\frac{\lambda_1}{K} - i\sigma^3 \frac{\pi\lambda_4\Omega^2}{2\lambda_1^4 K^2}\theta\left(-\frac{\Omega}{K}\right)\right] + \ldots, \\ \bar{v}_m &= \sigma a\left[\frac{\Omega}{K} \pm \frac{K}{\lambda_1} - i\sigma^3 \frac{\pi\lambda_4\Omega^3}{2\lambda_1^5 K^2}\theta\left(-\frac{\Omega}{K}\right)\right] + \\ &\quad \sigma^4 a Y_l\left[\lambda_1 - i\sigma^3 \frac{\pi\lambda_4\Omega^2}{2\lambda_1^4 K}\theta\left(-\frac{\Omega}{K}\right)\right] + \ldots \end{aligned} \tag{21}$$

The first of these formulae is derived from the equation of continuity $\bar{u}_m = -(iK)^{-1}d\bar{v}_m/dY_m$. In expressions (21) the main terms are retained separately for their real and imaginary parts which, having a different order of smallness with respect to the parameter σ, are nevertheless reliably calculated without referring to the leading terms in expansion (16).

From (4) and (6) we obtain the matching conditions for the Fourier-transforms $\bar{u}_l \to \bar{u}_m$, $\sigma\bar{v}_l \to \bar{v}_m$ at $Y_l \to \infty$, where functions with the subscript m are specified by Eqs.(21), and functions with subscript l are determined by expressions (14). These conditions lead to two equations in the Fourier-transform of the pressure $\bar{p}_l$ and the constant a. Eliminating the latter, we find

$$\bar{p}_l = \mp\Omega\bar{G}\left[\frac{\Omega}{K} \pm \frac{K}{\lambda_1} - i\sigma^3\frac{\pi\lambda_4\Omega^3}{2\lambda_1^5K^2}\theta\left(-\frac{\Omega}{K}\right) \mp i\sigma^3\frac{K^2}{(i\Omega)^{3/2}}\right]^{-1} \tag{22}$$

When deriving (22) we assumed $K = O(1)$. Formula (22) is unsuitable when $K = O(\sigma)$, since the phase velocity $c = -\sigma\Omega/K$ in this case is a value of the order of unity which violates the correctness of (20). The derivation of formula (22) is also ineffectual for $K = O(\sigma^{-1})$, since in this case the parameter $\delta = \sigma K$, by which the expansion (16) is carried out, is not small. Nevertheless, as follows from the laws of decay of expression (20) in the vicinity of zero and at infinity, its use over the whole real axis K, when taking the inverse Fourier transformation, introduces an error which does not exceed that (of order σ) resulting from expression (22) itself.

To be specific we will assume that $\Omega > 0$. Thus the critical layer exists when $K < 0$. Changing from Fourier transforms to the originals, for the pressure we obtain the formula

$$p_l' = \frac{1}{2\pi}\int\limits_{-\infty}^{0} \Omega\bar{G}\left[\frac{\Omega}{K} - \frac{K}{\lambda_1} - i\sigma^3\frac{\pi\lambda_4\Omega^3}{2\lambda_1^5K^2} - i\sigma^3\frac{K^2}{2^{1/2}\Omega^{3/2}}\right]^{-1}\times \tag{23}$$

$$\exp(iKX)\,dK - \frac{1}{2\pi}\int\limits_0^\infty \Omega\bar{G}\left[\frac{\Omega}{K} + \frac{K}{\lambda_1}\right]^{-1}\exp(iKX)\,dK$$

The integrand in the first of the integrals has a pole of the first order at the point $K^\circ = K_r^\circ + iK_i^\circ$ of the complex plane K, at a short distance of the order of σ^3 from the negative real semiaxis. If the real and imaginary parts of the denominator of the specified integrand are equated to zero we obtain

$$\begin{aligned} K_r^\circ &= -\lambda_1^{1/2}\Omega^{1/2} + O(\sigma) \\ K_i^\circ &= -\sigma^3\left(\frac{\pi\lambda_4\Omega^2}{4\lambda_1^5} + \frac{\lambda_1^2}{2^{3/2}\Omega^{1/2}}\right) + O(\sigma^4) \end{aligned} \tag{24}$$

The complex quantity K° is an eigenvalue of the wave number in the problem of free oscillations in the BL near to the upper branch of the NSC $(G = 0)$ and, in this case, expressions (24) makes up two parts of the dispersion relation. The condition $K_i^\circ = 0$ determines the frequency $\Omega = \Omega^\circ$ of neutral natural oscillations. From (24) we have

$$\Omega^\circ = (-2^{1/2}\pi^{-1}\lambda_1^7\lambda_4^{-1})^{2/5} \tag{25}$$

If we return to the initial system of units then Eq. (25) acquires the form $\omega^* = R^{2/5}U_\infty{}^*L^{*-1}\Omega^\circ$, which establishes the principal term of the asymptotic form of the upper branch of the NSC at large Reynolds numbers /11-13, 9, 10/.

We will consider the case $\Omega > \Omega^\circ$. Then, in accordance with (24) we have $K_i^\circ > 0$. Let the function $G'(X)$ differ from zero in the interval $0 < X < b$. Then when $X > b$ in the first term of formula (23), as a means of integration we replace the negative real semiaxis by a positive imaginary axis and an arc of a circle of fairly large radius in the second quadrant of the complex plane K, in which the integrand is exponentially small. As a result, expression (23) can be represented as the sum of the subtraction at the point K° and of two integrals along the imaginary positive and real semiaxes. Assuming that $S = iK$, we finally obtain

$$\begin{aligned} p_l' = &\frac{1}{2\pi}\int\limits_0^\infty \Omega\bar{G}(iS)\left(\frac{\Omega}{S} + \frac{S}{\lambda_1}\right)^{-1}\exp(-SX)\,dS - \\ &\frac{1}{2\pi}\int\limits_0^\infty \Omega\bar{G}(K)\left(\frac{\Omega}{K} + \frac{K}{\lambda_1}\right)^{-1}\exp(iKX)\,dK - \\ &{}^1/_2 i\lambda_1\Omega\bar{G}(K_r^\circ)\exp(iK_r^\circ X - K_i^\circ X) \end{aligned} \tag{26}$$

where K_r° and K_i° are defined by formulae (24). The final item in (26) generated by the pole of the Fourier transformant $\bar{p}_l$ is a TS wave with a decay decrement K_i°. Both integrals in (26) decay as $X \to \infty$ according to the law $O(X^{-2})$, whereas the exponential decrease of the amplitude of the TS wave downstream becomes as small as desired as the frequency Ω tends towards the neutral value Ω° from above.

By replacing the path of integration in the first integral of formula (23) by the negative imaginary semiaxis, it is possible to represent the pressure for the region upstream of the oscillator $(X < 0)$ in the form of the sum of two integral terms similar to the two first terms in (26) and possessing the asymptotic form $|X|^{-2}$, $X \to -\infty$. However, there is no term linked to the subtraction at the point K°.

We note that expression (26) does not retain special features at the point $\Omega = \Omega^\circ$ and also gives the solution to the problem for $\Omega < \Omega^\circ$ /14/, although, as a result of exponential growth, it does not additionally satisfy the condition for the Fourier transformation to be applicable.

The TS wave in the neighbourhood of the upper NSC branch is characterized by the relation $K_i^\circ/K_r^\circ = O(\sigma^3)$. This neighbourhood in the unstable region $\Omega < \Omega^\circ$ is overlapped by the neighbourhood of the lower NSC branch when $\Omega = O(\sigma^3)$. Indeed, from (24) it follows that $K_i^\circ/K_r^\circ = O(1)$, and $K_i^\circ = O(\sigma^{3/2})$, and the frequency and wave number in dimensional variables are

$$\begin{aligned} \omega^* &= R^{2/5}U_\infty^*L^{*-1}\Omega = O(R^{1/4}U_\infty^*L^{*-1}), \\ k^* &= R^{9/20}L^{*-1}K^\circ = O(R^{3/8}L^{*-1}) \end{aligned}$$

which corresponds to the asymptotic form of the lower branch of the NSC or its neighbourhood.

REFERENCES

1. TERENT'YEV E.D., Calculation of the pressure in the linear problem of a vibrator in a supersonic boundary layer. PMM, 43, 6, 1979.

2. TERENT'YEV E.D., The linear problem of a vibrator in a subsonic boundary layer. PMM, 45, 6, 1981.
3. NEILAND V.YA., Theory of the separation of a laminar boundary layer in supersonic flow. Izv. Akad. Nauk SSSR, MZG, 4, 1969.
4. STEWARTSON K., On the flow near the trailing edge of a flat plate. Mathematika, I, 16, 31, 1969.
5. SMITH F.T., On the non-parallel flow stability of the Blasius boundary layer. Proc. Roy. Soc. London. A, 366, 1724, 1979.
6. ZHUK V.I. and RYZHOV O.S., Free interaction and stability of the boundary layer in an incompressible liquid. Dokl. Akad. Nauk SSSR, 253, 6, 1980.
7. TERENT'YEV E.D., The linear problem of a vibrator executing harmonic oscillations at supercritical frequencies in a subsonic boundary layer. PMM, 48, 2, 1984.
8. VAZOV V., Asymptotic expansions of the solutions of ordinary differential equations, Mir, Moscow, 1968.
9. ZHUK V.I. and RYZHOV O.S., On the asymptotic form of the solutions of the Orr-Sommerfeld equations specifying unstable oscillations with large Reynolds numbers. Dokl. Akad. Nauk SSSR, 268, 6, 1983.
10. ZHUK V.I., On the asymptotic form of the solutions of the Orr-Sommerfeld equations bordering two branches of the neutral curve. Izv. Akad. Nauk SSSR. MZG, 4, 1984.
11. LIN C.C., On the stability of two-dimensional parallel flows. III. Stability in a viscous fluid. Quart. Appl. Math. 3, 4, 1946.
12. BODONYI R.J. and SMITH F.T., The upper branch stability of the Blasius boudary layer, including non-parallel flow effects. Proc. Roy. Soc. London, A, 375, 1760, 1981.
13. MIKHAILOV V.V., On the asymptotic form of the neutral curves of the linear problem of the stability of the laminar boundary layer. Izv. Akad. Nauk SSSR, MZG, 5, 1981.
14. BOGDANOVA E.V. and RYZHOV O.S., On perturbations generated by oscillators in the flow of a viscous liquid at critical frequencies. Prikl. Mat. Tekh. Fiz., 4, 1982.

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SEPARATION OF A FLOW FROM THE CORNER POINT OF A BODY*

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Changes in the velocity field which occur when the external pressure gradient is gradually increased, the gradient being determined by the theory of jet flows of an ideal incompressible fluid, are studied. The possibility of an essentially non-linear viscous sublayer occuring in the preseparation region, which adheres to the rigid surface, is noted. A solution of the boundary value problem is given for a boundary layer interacting freely with the potential flow under the conditions when the initial pressure gradient changes its sign from negative to positive. In this case a stagnation point appears in the incoming flow.

1. The preseparation region.

We shall assume that the surface of the streamlined body has a corner, at which the flow becomes separated. We choose the radius of curvature of the surface, the velocity of potential flow of fluid at the corner point, and its density, as the three basic units of measurement. Assuming that a change to dimensionless variables has been made, we shall direct the s axis of the curvilinear orthogonal system of coordinates along the generatrix of the body, and the n axis along its normal. Let u' and v' be the components of the perturbed velocity vectors and p' the excess pressure in the outer potential

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